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THE SOLUTION OF A CERTAIN NONLINEAR RIEMANN-HILBERT PROBLEM WITH AN APPLICATION

A. S. Peters

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Abstract

This report shows that the nonlinear barrier equation

$$F^+(\zeta)F^-(\zeta) + \mu(\zeta)[F^+(\zeta) + F^-(\zeta)] + \sigma(\zeta) = 0$$

can be solved in closed form provided $\mu(\zeta)$, $\sigma(\zeta)$ are Hölder continuous and so related that

$$\sigma(\zeta) + s^2(\zeta) + 2s(\zeta)\mu(\zeta) = [(\zeta-\alpha)(\zeta-\beta)]^\kappa t^2(\zeta)$$

where $s(\zeta)$ and $t(\zeta)$ are rational functions and $\kappa = 0$, or 1. It also contains a formula for the solution of

$$a(\zeta) \int_L \frac{\phi(z)dz}{z - \zeta} + \int_L \frac{b(z)\phi(z)dz}{z - \zeta} = f(\zeta)$$

if $a(\zeta)b(\zeta) = -1$; $a(\zeta) \neq 1, -1$.

1. Introduction

If $F(w)$ is a meromorphic function in the complex w-plane minus the simple, smooth path L from α to β, then, as is well known, the linear barrier equation

$$F^+(\zeta) + \mu(\zeta)F^-(\zeta) + \sigma(\zeta) = 0 \tag{1.1}$$

can be solved in closed form if $\mu(\zeta)$ and $\sigma(\zeta)$ are Hölder continuous, with $\mu(\zeta) \neq 0$, when ζ denotes a point on $L - \alpha - \beta$. The equation (1.1) has been studied intensively but, for more or less obvious reasons, comparatively little attention has been devoted to nonlinear barrier equations although Pogorzelski [1], in a series of papers, has obtained certain basic theoretical results for the equation

$$F^+(\zeta) + \nu(\zeta)F^-(\zeta) = \Phi[\zeta, F^+(\zeta), F^-(\zeta)] \ . \tag{1.2}$$

Equations of this type, for example

$$F^+(\zeta)F^-(\zeta) + \mu(\zeta)[F^+(\zeta) + F^-(\zeta)] + \sigma(\zeta) = 0 \ , \tag{1.3}$$

cannot be solved by performing only a finite number of operations if the coefficients are arbitrary Hölder continuous functions. However, the following sections show that if

$$\sigma(\zeta) + s^2(\zeta) + 2s(\zeta)\mu(\zeta) = [(\zeta-\alpha)(\zeta-\beta)]^\kappa t^2(\zeta) \ , \tag{1.4}$$

where κ is zero or one and $s(\zeta)$ and $t(\zeta)$ are rational functions, then the solution of (1.3) can be reduced to the solution of a linear

barrier equation. Some consequences of this are examined in Section 4 where we demonstrate that, subject to certain constraints on the coefficients, the equations

$$a(\zeta)\int_L \frac{\phi(z)dz}{z-\zeta} + \int_L \frac{b(z)\phi(z)dz}{z-\zeta} = f(\zeta)$$

and

$$a(\zeta)\oint_C \frac{\phi(z)dz}{z-\zeta} + \oint_C \frac{b(z)\phi(z)dz}{z-\zeta} = \pi i r(\zeta)\phi(\zeta) + f(\zeta)$$

can be solved in closed form.

2. Formulation

Suppose that L is a simple, smooth path directed from the initial point α to the terminal point β. In other words, suppose that the path L in the complex z-plane is defined by

$$z(\tau) = x(\tau) + iy(\tau) , \qquad \tau_o \leq \tau \leq \tau_e$$

where

$$z(\tau_1) \neq z(\tau_2) , \qquad \tau_o \leq \tau_1 \neq \tau_2 \leq \tau_e ;$$

the derivative $z'(\tau)$ is continuous in the closed interval $\tau_o \leq \tau \leq \tau_e$ such that $z'(\tau) \neq 0$; and the path is directed in accordance with increasing values of the parameter τ. The unit tangent vector to the path L is determined by $z'(\tau)/|z(\tau)|$ and if this is rotated counter-clockwise $\pi/2$ radians we obtain the unit normal which by definition points toward the positive side of L. Let the exterior of L, that is, the complete complex z-plane minus L, be denoted by D. Let $F(w)$ be a complex function defined for w in D and let $F^+(\zeta)$, $F^-(\zeta)$ denote the respective limit values of $F(w)$ as w approaches ζ, a point on $L' = L - \alpha - \beta$, from the positive side and negative side of L. Let us require the limit values $F^+(\zeta)$ and $F^-(\zeta)$ to be Hölder continuous on L' and, with the possible exception of a finite number of poles each of finite order, let us require $F(w)$ to be analytic when w is in D. Also, let us admit possible singularities at α and β to the extent that either $\int_L F^+(\zeta)d\zeta$ and $\int_L F^-(\zeta)d\zeta$ exist, or else $F(w)$ possesses a pole at one or both of the endpoints of L. Assume that each $c_i(\zeta)$ is a function which is Hölder continuous on L' and consider the

problem of finding $F(w)$ such that

$$c_1(\zeta)F^+(\zeta)F^-(\zeta) + c_2(\zeta)F^+(\zeta) + c_3(\zeta)F^-(\zeta) + c_4(\zeta) = C \ . \tag{2.1}$$

This poses a nonlinear Riemann-Hilbert problem. Our aim is to discuss some cases of (2.1) for which $F(w)$ can be obtained in closed form, that is, in terms of a finite number of quadratures.

The barrier equation (2.1) arises in connection with the solution of certain singular integral equations with Cauchy kernels. For example, the analysis of the nonlinear integral equation

$$\pi^2\phi^2(\zeta) + \left[\int_L \frac{\phi(z)dz}{z-\zeta}\right]^2 + 2\mu(\zeta)\int_L \frac{\phi(z)dz}{z-\zeta} + \sigma(\zeta) = 0 \tag{2.2}$$

can be converted to the analysis of a barrier equation by introducing the function

$$F_1(w) = \int_L \frac{\phi(z)dz}{z-w} \ . \tag{2.3}$$

In equation (2.2) ζ denotes a point in L' and therefore it can be imbedded in a subarc L_ε of L such that the extremities of L_ε are the same distance ε from ζ. The integral symbol in (2.2) means the Cauchy principal value, namely,

$$\int_L \frac{\phi(z)dz}{z-\zeta} = \lim_{\varepsilon\to 0}\int_{L-L_\varepsilon} \frac{\phi(z)dz}{z-\zeta} \tag{2.4}$$

provided the limit exists. The existence of this limit is insured if we require $\phi(z)$ to be Hölder continuous on L'. Then, the Plemelj

formulas show that

$$F_1^+(\zeta) = \pi i\phi(\zeta) + \int_L \frac{\phi(z)dz}{z-\zeta}$$

(2.5)

$$F_1^-(\zeta) = -\pi i\phi(\zeta) + \int_L \frac{\phi(z)dz}{z-\zeta}$$

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or, by addition and subtraction

$$\int_L \frac{\phi(z)dz}{z-\zeta} = \frac{F_1^+(\zeta) + F_1^-(\zeta)}{2}$$

(2.6)

$$\phi(\zeta) = \frac{F_1^+(\zeta) - F_1^-(\zeta)}{2\pi i} .$$

It follows that if a solution of (2.2) exists, then $F_1(w)$ must satisfy

$$F_1^+(\zeta)F_1^-(\zeta) + \mu(\zeta)[F_1^+(\zeta) + F_1^-(\zeta)] + \sigma(\zeta) = 0 \tag{2.7}$$

where the limit values are Hölder continuous on L'. Here, $F_1(w)$ must be analytic in D and it must vanish at infinity in accordance with the integral representation of $F_1(w)$ in (2.3). The behavior of $F_1(w)$ at the endpoints of L depends upon the conditions imposed on $\phi(z)$ at α and β.

The barrier equation (2.1) can also be related to the simultaneous linear homogeneous barrier equations

$$\psi_1^+(\zeta) = c_{11}(\zeta)\psi_1^-(\zeta) + c_{12}(\zeta)\psi_2^-(\zeta) \tag{2.8}$$

$$\psi_2^+(\zeta) = c_{21}(\zeta)\psi_1^-(\zeta) + c_{22}(\zeta)\psi_2^-(\zeta) \tag{2.9}$$

where $\psi_1(w)$ and $\psi_2(w)$ are required to be of finite degree at infinity and analytic elsewhere in D. If we divide (2.8) by (2.9) and introduce

$$F(w) = \frac{\psi_1(w)}{\psi_2(w)}$$

we find

$$F^+(\zeta) = \frac{c_{11}(\zeta)F^-(\zeta) + c_{12}(\zeta)}{c_{21}(\zeta)F^-(\zeta) + c_{22}(\zeta)}$$

or

$$c_{21}(\zeta)F^+(\zeta)F^-(\zeta) + c_{22}(\zeta)F^+(\zeta) - c_{11}(\zeta)F^-(\zeta) - c_{12}(\zeta) = 0 . \tag{2.10}$$

Note that here, in contrast to (2.7), F(w) may have poles in D. The equation (2.10) with either (2.8) or (2.9) is equivalent to the system (2.8), (2.9). If we can find, in closed form, any admissible function which satisfies (2.10), then we can solve (2.8), (2.9) in closed form.

The analysis of (2.8), (2.9) is essential if we wish to study a pair of Cauchy singular integral equations, namely,

$$\sum_{j=1}^{2} \left[a_{ij}\phi_j(\zeta) + b_{ij} \int_L \frac{\phi_j(z)dz}{z-\zeta} \right] = f_i(\zeta) , \quad i = 1,2 . \tag{2.11}$$

Note also in passing that the equation

$$a(\zeta) \int_L \frac{\phi_1(z)dz}{z-\zeta} + \int_L \frac{b(z)\phi_1(z)dz}{z-\zeta} = f_1(\zeta) \tag{2.12}$$

can be regarded as defining a system of the type (2.11) if we write (2.12) in the form

$$a(\zeta) \int_L \frac{\phi_1(z)dz}{z-\zeta} + \int_L \frac{\phi_2(z)dz}{z-\zeta} = f_1(\zeta) ,$$

$$b(\zeta)\phi_1(\zeta) - \phi_2(\zeta) = 0 .$$

This implies that the solution of (2.12) can be reduced to the solution of the barrier equation (2.1). For additional relevant remarks and an analysis of the system (2.11), including some cases of (2.12), see Peters [2].

3. The Solution of Some Nonlinear Barrier Equations

It can be shown that, with $F(w)$ subject to the restrictions imposed in Section 2, the barrier equation

$$F^+(\zeta)F^-(\zeta) + \mu(\zeta)F^+(\zeta) + \nu(\zeta)F^-(\zeta) + \sigma(\zeta) = 0 \tag{3.1}$$

cannot be solved in closed form if the coefficients are arbitrary Hölder continuous functions. This means that the barrier equation (3.1) is not in general equivalent to a linear barrier equation. There are, however, some interesting cases for which (3.1) can be reduced, in a more or less simple way, to a linear problem. For example, consider

$$F^+(\zeta)F^-(\zeta) + k_1F^+(\zeta) + k_2F^-(\zeta) + k_3 = 0 \tag{3.2}$$

where the k_i's are constants. If we set

$$F(w) = \frac{1}{\underline{F}(w)} + k$$

where the constant k satisfies

$$k^2 + (k_1 + k_2)k + k_3 = 0 ,$$

then (3.2) becomes

$$(k + k_2)\underline{F}^-(\zeta) + (k + k_1)\underline{F}^-(\zeta) = -1 ;$$

a simple linear barrier equation whose closed solution is well known. As a second example, consider

(3.3) $$F^+(\zeta)F^-(\zeta) + r(\zeta)[F^+(\zeta) + F^-(\zeta)] + \sigma(\zeta) = 0$$

where $r(w)$ is a ratio of polynomials; while $r(\zeta)$ and $\sigma(\zeta)$ are Hölder continuous on L'. Equation (3.3) can be written

$$[F(\zeta) + r(\zeta)]^+[F(\zeta) + r(\zeta)]^- + \sigma(\zeta) - r^2(\zeta) = 0$$

and if we introduce

$$F(w) + r(w) = \exp G(w)$$

we have the linear equation

$$G^+(\zeta) + G^-(\zeta) = \ell n\ [r^2(\zeta) - \sigma(\zeta)] \ .$$

We turn next to some examples which are more general and a little more complicated.

Hereafter we are going to be concerned with

(3.4) $$F^+(\zeta)F^-(\zeta) + \mu(\zeta)[F^+(\zeta) + F^-(\zeta)] + \sigma(\zeta) = 0$$

when $\mu(\zeta)$ and $\sigma(\zeta)$ are Hölder continuous on L'; and so related that

(3.5) $$\sigma(\zeta) + s^2(\zeta) + 2s(\zeta)\mu(\zeta) = [(\zeta-\alpha)(\zeta-\beta)]^\kappa t^2(\zeta)$$

$$s(\zeta) = \frac{s_1(\zeta)}{s_2(\zeta)} \ ; \qquad t(\zeta) = \frac{t_1(\zeta)}{t_2(\zeta)}$$

where $s_1(\zeta)$, $s_2(\zeta)$, $t_1(\zeta)$, $t_2(\zeta)$ are polynomials and

$$\kappa = 0 \quad \text{or} \quad 1 \ .$$

Let us consider the case k = 1 first. If we set

$$F(w) = G_1(w) + s(w) + t(w)\sqrt{(w-\alpha)(w-\beta)} \tag{3.6}$$

we find by virtue of (3.5) that $G_1(w)$ must satisfy

$$G_-^+(\zeta)G_1^-(\zeta) + \left[s(\zeta)+\mu(\zeta) - t(\zeta)\sqrt{(\zeta-\alpha)(\zeta-\beta)}\right]G_1^+(\zeta) + \left[s(\zeta)+\mu(\zeta)+t(\zeta)\sqrt{(\zeta-\alpha)(\zeta-\beta)}\right]G_1^-(\zeta) = 0 . \tag{3.7}$$

Therefore if we set

$$G_1(w) = H_1(w)I_1(w)$$

where

$$H_1(w) = \exp \frac{1}{2\pi i}\int_L \frac{\gamma_1(z)dz}{z-w}$$

and

$$\gamma_1(z) = \ell n \left[\frac{t(z)\sqrt{(z-\alpha)(z-\beta)} + s(z) + \mu(z)}{t(z)\sqrt{(z-\alpha)(z-\beta)} - s(z) - \mu(z)}\right]$$

varies continuously along L, we find that $I_1(w)$ must satisfy

$$\frac{1}{I_1^+(\zeta)} - \frac{1}{I_1^-(\zeta)} = -\frac{H_1(\zeta)}{\sqrt{t^2(\zeta)(\zeta-\alpha)(\zeta-\beta) - [s(\zeta)+\mu(\zeta)]^2}} . \tag{3.8}$$

The general solution of (3.8) is

$$\frac{1}{I_1(w)} = p(w) - \frac{1}{2\pi i}\int_L \frac{H_1(z)dz}{(z-w)\sqrt{t^2(z)(z-\alpha)(z-\beta) - [s(z)+\mu(z)]^2}} \tag{3.9}$$

where

(3.10) $$p^+(\zeta) = p^-(\zeta) = p(\zeta) .$$

From this relation and the conditions imposed on F(w) we deduce that p(w) must be a ratio of polynomials. Thus we have shown that the solution of (3.4) can be presented in closed form if (3.5) holds with $\kappa = 1$.

The equation (3.4) subject to (3.5) with $\kappa = 0$, i.e.,

(3.11) $$\sigma(\zeta) + s^2(\zeta) + 2s(\zeta)\mu(\zeta) = t^2(\zeta)$$

can also be solved in closed form. To see this, let

(3.12) $$F(w) = G(w) + s(w) + t(w) .$$

The substitution of (3.12) in (3.4) subject to (3.11) shows that G(w) must satisfy

$$G^+(\zeta)G^-(\zeta) + [s(\zeta) + t(\zeta) + \mu(\zeta)][G^+(\zeta) + G^-(\zeta)]$$

$$+ 2t(\zeta)[s(\zeta) + t(\zeta) + \mu(\zeta)] = 0$$

or

$$\frac{1}{G^+(\zeta)G^-(\zeta)} + \frac{1}{2t(\zeta)}\left[\frac{1}{G^+(\zeta)} + \frac{1}{G^-(\zeta)}\right] + \frac{1}{2t(\zeta)[s(\zeta) + t(\zeta) + \mu(\zeta)]} = 0 .$$

The last equation can be written

(3.13) $$\left[\frac{2t(\zeta)}{G(\zeta)} + 1\right]^+\left[\frac{2t(\zeta)}{G(\zeta)} + 1\right]^- = \frac{\mu(\zeta) + s(\zeta) - t(\zeta)}{\mu(\zeta) + s(\zeta) + t(\zeta)} .$$

If we assume that

$$N(\zeta) = \frac{\mu(\zeta) + s(\zeta) - t(\zeta)}{\mu(\zeta) + s(\zeta) + t(\zeta)}$$

neither vanishes nor becomes infinite on L' and if we define $\ell n\ N(z)$ so that it varies continuously as z changes from α to β along L, then the general solution of (3.13) is

$$\frac{2t(w)}{G(w)} + 1 = e^{h(w)+q(w)} \tag{3.14}$$

where

$$h(w) = \frac{\sqrt{(w-\alpha)(w-\beta)}}{2\pi i} \int_L \frac{\ell n\ N(z)dz}{(z-w)\sqrt{(z-\alpha)(z-\beta)}} \tag{3.15}$$

and q(w) must obey

$$q^+(\zeta) + q^-(\zeta) = 0 .$$

Hence we find that

$$\begin{aligned} F(w) &= s(w) + t(w) + G(w) \\ &= s(w) + t(w)\, \frac{e^{h(w)+q(w)} + 1}{e^{h(w)+q(w)} - 1} . \end{aligned} \tag{3.16}$$

If $F_o(w)$ is any admissible function such that

$$F_o^+(\zeta)F_o^-(\zeta) + \mu(\zeta)[F_o^+(\zeta) + F_o^-(\zeta)] + \sigma(\zeta) = 0 \tag{3.17}$$

then the general solution of

$$F^+(\zeta)F^-(\zeta) + \mu(\zeta)[F^+(\zeta) + F^-(\zeta)] + \sigma(\zeta) = 0 \tag{3.18}$$

can be expressed in terms of $F_o(w)$ as follows. Take

$$F(w) = F_o(w) + F_1(w) . \tag{3.19}$$

Substitution in (3.18) shows that $F_1(w)$ must satisfy

$$\frac{\mu(\zeta)+F_o^+(\zeta)}{F_1^+(\zeta)} + \frac{\mu(\zeta)+F_o^-(\zeta)}{F_1^-(\zeta)} + 1 = 0 \ . \tag{3.20}$$

This is a linear barrier equation which can easily be solved for $1/F_1(w)$; and hence we can show explicitly how $F_1(w)$ depends on $F_o(w)$.

If

$$\mu(\zeta) = r(\zeta) \ ,$$

where $r(w)$ is a rational function, then the solution of (3.20) can be written immediately because we have

$$\left[\frac{r(\zeta)+F_o(\zeta)}{F_1(\zeta)} + \frac{1}{2}\right]^+ + \left[\frac{r(\zeta)+F_o(\zeta)}{F_1(\zeta)} + \frac{1}{2}\right]^- = 0$$

from which

$$\frac{r(w)+F_o(w)}{F_1(w)} + \frac{1}{2} = q_1(w) \tag{3.21}$$

where

$$q^+(\zeta)+q_1^-(\zeta) = 0 \ .$$

From (3.21)

$$F_1(w) = \frac{2[r(w)+F_o(w)]}{2q_1(w)-1} \ ,$$

so that the general solution of (3.18) is

$$F(w) = F_o(w) + \frac{2[r(w)+F_o(w)]}{2q_1(w)-1}$$

$$= -r(w) + \frac{2q_1(w)+1}{2q_1(w)-1} \cdot \left[r(w)+F_o(w)\right]$$

or

$$F(w) = -r(w) + \rho(w)[r(w) + F_o(w)] \tag{3.22}$$

where

$$\rho^+(\zeta)\rho^-(\zeta) = 1 .$$

If the condition (3.5) holds we see from

$$F_o^+(\zeta)F_o^-(\zeta) + \mu(\zeta)[F_o^+(\zeta) + F_o^-(\zeta)]$$

$$+ [(\zeta-\alpha)(\zeta-\beta)]^{\kappa}t^2(\zeta) - s^2(\zeta) - 2s(\zeta)\mu(\zeta) = 0$$

that

$$\mu(\zeta) = \frac{s^2(\zeta) - [(\zeta-\alpha)(\zeta-\beta)]^{\kappa}t^2(\zeta) - F_o^+(\zeta)F_o^-(\zeta)}{F_o^+(\zeta) + F_o^-(\zeta) - 2s(\zeta)} ;$$

$$\mu(\zeta) + F_o^+(\zeta) = \frac{[F_o^+(\zeta) - s(\zeta)]^2 - [(\zeta-\alpha)(\zeta-\beta)]^{\kappa}t^2(\zeta)}{F_o^+(\zeta) + F_o^-(\zeta) - 2s(\zeta)} ;$$

and

$$\mu(\zeta) + F_o^-(\zeta) = \frac{[F_o^-(\zeta) - s(\zeta)]^2 - [(\zeta-\alpha)(\zeta-\beta)]^{\kappa}t^2(\zeta)}{F^+(\zeta) + F_o^-(\zeta) - 2s(\zeta)} .$$

Hence (3.20) can be written

$$\left\{ \frac{[F_o(\zeta) - s(\zeta)]^2 - [(\zeta-\alpha)(\zeta-\beta)]^{\kappa}t^2(\zeta)}{F_1(\zeta)} + F_o(\zeta) - s(\zeta) \right\}^+$$

$$+ \left\{ \frac{[F_o(\zeta) - s(\zeta)]^2 - [(\zeta-\alpha)(\zeta-\beta)]^{\kappa}t^2(\zeta)}{F_1(\zeta)} + F_o(\zeta) - S(\zeta) \right\}^- = 0 .$$

The general solution of the last equation is

$$(3.23)\qquad \frac{[F_o(w)-s(w)]^2-[(w-\alpha)(w-\beta)]^{\kappa}t^2(w)}{F_1(w)}+F_o(w)-s(w)=\frac{1}{q_2(w)}$$

where

$$q_2^+(\zeta)+q_2^-(\zeta)=0\ .$$

Therefore

$$F_1(w)=q_2(w)\,\frac{\{[F_o(w)-s(w)]^2-[(w-\alpha)(w-\beta)]^{\kappa}t^2(w)\}}{1-q_2(w)[F_o(w)-s(w)]}\ .$$

This gives

$$(3.24)\qquad F(w)=s(w)+\frac{[F_o(w)-s(w)]-q_2(w)[(w-\alpha)(w-\beta)]^{\kappa}t^2(w)}{1-q_2(w)[F_o(w)-s(w)]}$$

which is a representation for the general solution of (3.18) subject to (3.5) where $\kappa = 0,\ 1$.

The basic procedures presented above can also be used for a closed curve but some modifications are needed. Let C be a simple smooth closed path which divides the complex plane into two domains D^+ and D^- of which D^+ contains the positive side of C. Let us seek a function F(w) which, except for poles, is analytic in D^+ and D^-; and which satisfies the barrier equation

$$(3.25)\qquad F^+(\zeta)F^-(\zeta)+\mu(\zeta)[F^+(\zeta)+F^-(\zeta)]+r(\zeta)[F^+(\zeta)-F^-(\zeta)]+\sigma(\zeta)=0$$

for ζ on C. Let us assume that r(w) is a rational function; that each of $\mu(\zeta)$, $\sigma(\zeta)$ and $r(\zeta)$ is Hölder continuous on C; and that the coefficients are so related that

$$(3.26)\qquad \sigma(\zeta)+r^2(\zeta)+s^2(\zeta)+2s(\zeta)\mu(\zeta)=t^2(\zeta)$$

where $s(\zeta)$ and $t(\zeta)$ are rational functions. With (3.26) the barrier equation (3.25) can be solved in closed form as follows. If we substitute

$$F(w) = \begin{cases} G(w) + s(w) + t(w) + r(w) & w \text{ in } D^+ \\ G(w) + s(w) + t(w) - r(w) & w \text{ in } D^- \end{cases} \tag{3.27}$$

in (3.25) we find that $G(w)$ must satisfy

$$\frac{1}{G^+(\zeta)G^-(\zeta)} + \frac{1}{2t(\zeta)}\left[\frac{1}{G^+(\zeta)} + \frac{1}{G^-(\zeta)}\right] + \frac{1}{2t(\zeta)[s(\zeta)+t(\zeta)+\mu(\zeta)]} = 0$$

which can be expressed as

$$\left[\frac{2t(\zeta)}{G(\zeta)}+1\right]^+\left[\frac{2t(\zeta)}{G(\zeta)}+1\right]^- = \frac{\mu(\zeta)+s(\zeta)-t(\zeta)}{\mu(\zeta)+s(\zeta)+t(\zeta)} \,. \tag{3.28}$$

This can be solved by taking the logarithm of both sides but before doing this an adjustment may be necessary. Assuming that

$$N(\zeta) = \frac{\mu(\zeta)+s(\zeta)-t(\zeta)}{\mu(\zeta)+s(\zeta)+t(\zeta)}$$

neither vanishes nor becomes infinite for ζ on C, it may be that the argument of $\ell n\, N(\zeta)$ changes by $m \cdot 2\pi i$, an integral multiple of $2\pi i$, as ζ ranges along C back to a starting point and so $N(\zeta)$ may be discontinuous on C. In order to compensate for this we can introduce the factor $(\zeta - w_o)^m$, where w_o is in D^+, as shown in the next equation

$$\left[\frac{2t(\zeta)}{G(\zeta)}+1\right]^+\left[\frac{2t(\zeta)}{G(\zeta)}+1\right]^- \cdot \frac{1}{(\zeta-w_o)^m} = \frac{N(\zeta)}{(\zeta-w_o)^m} \,. \tag{3.29}$$

The general solution of the last equation is

$$\frac{2t(w)}{G(w)} + 1 = \begin{cases} (w-w_o)^m \exp\,[h_o(w) + p(w)] & w \text{ in } D^+ \\ \exp\,[-h_o(w) - p(w)] & w \text{ in } D^- \end{cases}$$

where

$$h_o(w) = \frac{1}{2\pi i} \oint_C \frac{\ell n \left[\dfrac{N(z)}{(z-w_o)^m}\right]}{z - w}\, dz$$

and $p(w)$ is such that

$$p^+(\zeta) = p^-(\zeta) = p(\zeta) \ .$$

From the above

$$G(w) = \begin{cases} \dfrac{2t(w)}{(w-w_o)^m e^{h_o(w)+p(w)} - 1} & w \text{ in } D^+ \\ \dfrac{2t(w)}{e^{-h_o(w)-p(w)} - 1} & w \text{ in } D^- \end{cases}$$

and we find

$$F(w) = \begin{cases} s(w) + r(w) + t(w) \cdot \dfrac{(w-w_o)^m e^{h_o(w)+p(w)} + 1}{(w-w_o)^m e^{h_o(w)+p(w)} - 1} & w \text{ in } D^+ \\ s(w) - r(w) + t(w) \cdot \dfrac{e^{-h_o(w)-p(w)} + 1}{e^{-h_o(w)-p(w)} - 1} & w \text{ in } D^- \end{cases} \tag{3.30}$$

as the general solution of (3.25) subject to (3.26). The integer m is the index of $N(z)$ on C and it is given by

$$m = \frac{1}{2\pi i} \oint_C \frac{N'(z)}{N(z)}\, dz \ .$$

4. An Application

Let us apply some of the preceding ideas to the equation

$$a(\zeta)\int_L \frac{\phi(z)dz}{z-\zeta} + \int_L \frac{b(z)\phi(z)dz}{z-\zeta} = f(\zeta)\ . \tag{4.1}$$

We proceed to demonstrate that if

$$\begin{aligned} &a(\zeta)b(\zeta) = -1 \\ &a(\zeta) \neq 1\ ,\ -1\ ; \end{aligned} \tag{4.2}$$

then the solution of (4.1) can be presented in closed form because the analysis of (4.1) can be reduced to the analysis of a linear barrier equation. We assume that $a(\zeta)$, $b(\zeta)$, $f(\zeta)$ and $\phi(\zeta)$ are integrable along L; and that they are Hölder continuous when ζ denotes a point in $L' = L - \alpha - \beta$.

In order to solve (4.1) introduce

$$\chi(w) = \chi_1(w)\int_L \frac{\phi(z)dz}{z-w} + \chi_2(w)\int_L \frac{b(z)\phi(z)dz}{z-w} \tag{4.3}$$

where $\chi_1(w)$ and $\chi_2(w)$ are required to be analytic in D except for possible poles at infinity. The Plemelj formulas, with (4.1), show that if we require $\chi_1(w)$ and $\chi_2(w)$ to be two functions, not both identically zero, which satisfy the simultaneous linear barrier equations

$$\chi_1^+(\zeta) + b(\zeta)\chi_2^+(\zeta) = -[\chi_1^-(\zeta) + b(\zeta)\chi_2^-(\zeta)]\ ; \tag{4.4}$$

$$\chi_1^+(\zeta) - a(\zeta)\chi_2^+(\zeta) = \chi_1^-(\zeta) - a(\zeta)\chi_2^-(\zeta)\ ; \tag{4.5}$$

then $\chi(w)$ must satisfy the equation

$$\chi^+(\zeta) - \chi^-(\zeta) = [\chi_2^+(\zeta) - \chi_2^-(\zeta)]f(\zeta) \ . \tag{4.6}$$

We also have

$$\begin{aligned}\chi^+(\zeta) + \chi^-(\zeta) = {} & 2\pi i\{\chi_1^+(\zeta) + b(\zeta)\chi_2^+(\zeta)\}\phi(\zeta) \\ & + 2\{\chi_1^+(\zeta) - a(\zeta)\chi_2^+(\zeta)\}\int_L \frac{\phi(z)dz}{z-\zeta} \\ & + \{\chi_2^+(\zeta) + \chi_2^-(\zeta)\}f(\zeta) \ .\end{aligned} \tag{4.7}$$

It is now evident that if $\chi_1(w)$ and $\chi_2(w)$ can be found, then (4.6) can be solved for $\chi(w)$ and hence $\phi(z)$ can be deduced from (4.7) by introducing

$$\Omega(w) = \int_L \frac{\phi(z)dz}{z-w} \tag{4.8}$$

which provides

$$2\pi i\ \phi(\zeta) = \Omega^+(\zeta) - \Omega^-(\zeta) \tag{4.9}$$

and

$$2\int_L \frac{\phi(z)dz}{z-\zeta} = \Omega^+(\zeta) + \Omega^-(\zeta) \ . \tag{4.10}$$

In terms of $\Omega(w)$ the equation (4.7) becomes

$$\begin{aligned}\chi^+(\zeta) + \chi^-(\zeta) = {} & \{2\chi_1^+(\zeta) + [b(\zeta) - a(\zeta)]\chi_2^+(\zeta)\}\Omega^+(\zeta) \\ & - [b(\zeta) + a(\zeta)]\chi_2^+(\zeta)\Omega^-(\zeta) + \{\chi_2^+(\zeta) + \chi_2^-(\zeta)\}f(\zeta)\end{aligned} \tag{4.11}$$

and if we use (4.4) and (4.5) we have

$$\{2\chi_1^+(\zeta) + [b(\zeta) - a(\zeta)]\chi_2^+(\zeta)\}\Omega^+(\zeta)$$

$$(4.12) \qquad + \{2\chi_1^-(\zeta) + [b(\zeta) - a(\zeta)]\chi_2^-(\zeta)\}\Omega^-(\zeta)$$

$$= \chi^+(\zeta) + \chi^-(\zeta) - \{\chi_2^+(\zeta) + \chi_2^-(\zeta)\} f(\zeta) ,$$

a linear barrier equation which can be solved for $\Omega(w)$ once $\chi_1(w)$ and $\chi_2(w)$ are known. The equation (4.12) can be solved if we know only certain particular solutions of (4.4), (4.5) and (4.6).

The discovery of admissible solutions of the system (4.4), (4.5) can be made to depend on a solution of the nonlinear barrier equation for

$$(4.13) \qquad F(w) = \frac{\chi_1(w)}{\chi_2(w)}$$

which is generated by dividing (4.4) by (4.5). We find

$$(4.14) \qquad F^+(\zeta)F^-(\zeta) + \frac{[b(\zeta) - a(\zeta)]}{2} [F^+(\zeta) + F^-(\zeta)] + 1 = 0 .$$

The pair (4.14) and either (4.4) or (4.5), is equivalent to (4.4) and (4.5).

Since (4.2) is in force, we know from Section 3 and (3.16) that if

$$(4.15) \qquad \lambda(w) = \frac{\sqrt{(w-\alpha)(w-\beta)}}{2\pi i} \int_L \frac{\ell n \left[\frac{1 + a(z)}{1 - a(z)}\right]}{(z-w)\sqrt{(z-\alpha)(z-\beta)}} \, dz ,$$

where $[1 + a(z)]/[1 - a(z)]$ neither vanishes nor becomes infinite for z on L, then

(4.16) $$F_1(w) = \frac{e^{2\lambda(w)} + 1}{e^{2\lambda(w)} - 1} = \coth \lambda(w)$$

is a particular solution of (4.14). From this, we find by substituting

$$\chi_1(w) = F_1(w)\chi_2(w)$$

in (4.4) and (4.5), that $\chi_2(w)$ must

(4.17) $$\left\{F_1^+(\zeta) + \frac{[b(\zeta) - a(\zeta)]}{2}\right\}[\chi_2^+(\zeta)]^2 - \left\{F_1^-(\zeta) + \frac{[b(\zeta) - a(\zeta)]}{2}\right\}[\chi_2^-(\zeta)]^2 = 0$$

Equation (4.14) shows that

(4.18) $$\frac{b(\zeta) - a(\zeta)}{2} = -\frac{[F_1^+(\zeta)F_1^-(\zeta) + 1]}{F_1^+(\zeta) + F_1^-(\zeta)}$$

and therefore equation (4.17) is the same as

(4.19) $$\{[F_1^+(\zeta)]^2 - 1\}[\chi_2^+(\zeta)]^2 - \{[F_1^-(\zeta)]^2 - 1\}[\chi_2^-(\zeta)]^2 = 0 .$$

The function

(4.20) $$\chi_2(w) = \frac{1}{\sqrt{F_1^2(w) - 1}} = \sinh \lambda(w)$$

satisfies (4.19); it is analytic in D and behaves like a constant at infinity. Corresponding to this $\chi_2(w)$, the function $\chi_1(w)$ is

(4.21) $$\chi_1(w) = F_1(w)\chi_2(w) = \cosh \lambda(w)$$

and it also satisfies the original conditions imposed on χ_1 and χ_2 by being analytic in D and behaving like a constant at infinity.

Notice that if we define

$$v(\zeta) = \frac{1}{2} \ell n \left[\frac{1+a(\zeta)}{1-a(\zeta)}\right] , \tag{4.22}$$

and

$$\lambda(\zeta) = \frac{\sqrt{(\zeta-\alpha)(\zeta-\beta)}}{2\pi i} \int_L \frac{\ell n \left[\frac{1+a(z)}{1-a(z)}\right]}{(z-\zeta)\sqrt{(z-\alpha)(z-\beta)}} dz ; \tag{4.23}$$

then

$$\lambda^+(\zeta) = v(\zeta) + \lambda(\zeta)$$

and

$$\lambda^-(\zeta) = v(\zeta) - \lambda(\zeta) .$$

Notice also that (4.6) and (4.20) yield

$$\chi^+(\zeta) - \chi^-(\zeta) = \left[e^{v(\zeta)} + e^{-v(\zeta)}\right] f(\zeta) \sinh \lambda(\zeta)$$

$$= \frac{2f(\zeta) \sinh \lambda(\zeta)}{\sqrt{1-a^2(\zeta)}} .$$

Hence

$$\chi(w) = \frac{1}{\pi i} \int_L \frac{f(z) \sinh \lambda(z)}{(z-w)\sqrt{1-a^2(z)}} \tag{4.24}$$

is the only solution of (4.6) with properties that match those of the representation (4.3).

We are now in a position to solve (4.12) for $\Omega(w)$. If we use (4.13), (4.18) and (4.6), then (4.12) becomes

$$\{[F_1^+(\zeta)]^2 - 1\}\chi_2^+(\zeta)\Omega^+(\zeta) + \{[F_1^-(\zeta)]^2 - 1\}\chi_2^-(\zeta)\Omega^-(\zeta)$$

$$= F_1^+(\zeta)\{\chi^+(\zeta) - \chi_2^+(\zeta)f(\zeta)\} + F_1^-(\zeta)\{\chi^-(\zeta) - \chi_2^-(\zeta)f(\zeta)\} .$$

This leads to

$$\left\{\frac{\Omega(w)}{\sinh \lambda(w)} - \chi(w) \coth \lambda(w)\right\}^+ + \left\{\frac{\Omega(w)}{\sinh \lambda(w)} - \chi(w) \coth \lambda(w)\right\}^- = -\frac{2f(\zeta) \cosh \lambda(\zeta)}{\sqrt{1 - a^2(\zeta)}} .$$

Subject to the implications of the representation (4.8), the general solution of the last equation is

$$(4.25)\quad \Omega(w) = \begin{cases} \dfrac{\cosh \lambda(w)}{\pi i} \displaystyle\int_L \dfrac{f(z) \sinh \lambda(z)}{(z-w)\sqrt{1 - a^2(z)}}\, dz \\ \dfrac{-\sinh \lambda(w)}{\pi i\sqrt{(w-\alpha)(w-\beta)}} \displaystyle\int_L \dfrac{\sqrt{(z-\alpha)(z-\beta)}\, f(z) \cosh \lambda(z)}{(z-w)\sqrt{1 - a^2(z)}}\, dz \\ + \dfrac{\pi i c \sinh \lambda(w)}{\sqrt{(w-\alpha)(w-\beta)}} \end{cases}$$

where c is an arbitrary constant.

Finally, the solution of the integral equation (4.1), subject to (4.2), is determined by (4.9) and (4.25). The formula for the solution is

$$(4.26)\quad \phi(\zeta) = -\frac{a(\zeta)\sinh\lambda(\zeta)}{\pi^2\sqrt{1-a^2(\zeta)}}\int_L \frac{f(z)\sinh\lambda(z)}{(z-\zeta)\sqrt{1-a^2(z)}}\,dz$$

$$+\frac{a(\zeta)\cosh\lambda(\zeta)}{\pi^2\,(\zeta-\alpha)(\zeta-\beta)\sqrt{1-a^2(\zeta)}}\int_L \frac{\sqrt{(z-\alpha)(z-\beta)}\,f(z)\cosh\lambda(z)}{(z-\zeta)\sqrt{1-a^2(z)}}\,dz$$

$$+\frac{c\,a(\zeta)\cosh\lambda(\zeta)}{\sqrt{(\zeta-\alpha)(\zeta-\beta)}\,\sqrt{1-a^2(\zeta)}}$$

where $\lambda(\zeta)$ is defined by (4.23).

There are other equations of the type (4.1), namely

$$(4.27)\qquad a(\zeta)\int_L \frac{\phi(z)dz}{z-\zeta} + \int_L \frac{b(z)\phi(z)dz}{z-\zeta} = f(\zeta)\ ,$$

which yield to the above procedures. The condition (4.2) is a particular case of

$$(4.28)\quad s^2(\zeta)+s(\zeta)[b(\zeta)-a(\zeta)] = a(\zeta)b(\zeta)+[(\zeta-\alpha)(\zeta-\beta)]^{\kappa}t^2(\zeta)$$

where $s(\zeta)$ and $t(\zeta)$ are rational functions and κ is zero or one. It can be verified that if a solution of (4.27), subject to (4.28), exists, then it can be presented in closed form. The same is true for the equation

$$(4.29)\qquad a(\zeta)\oint_C \frac{\phi(z)dz}{z-\zeta} + \oint_C \frac{b(z)\phi(z)dz}{z-\zeta} = r(\zeta)\phi(\zeta)+f(\zeta)$$

when its coefficients obey the relation

(4.30) $$s^2(\zeta) + s(\zeta)[b(\zeta) - a(\zeta)] = a(\zeta)b(\zeta) + r^2(\zeta) + t^2(\zeta)$$

where r(s) is a rational function.

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This report shows that the nonlinear barrier equation

$$F^+(\zeta)F^-(\zeta)+\mu(\zeta)[F^+(\zeta)+F^-(\zeta)]+\sigma(\zeta)=0$$

can be solved in closed form provided $\mu(\zeta)$, $\sigma(\zeta)$ are Hölder continuous and so related that

$$\sigma(\zeta)+s^2(\zeta)+2s(\zeta)\mu(\zeta)=[(\zeta-\alpha)(\zeta-\beta)]^{\kappa}t^2(\zeta)$$

where $s(\zeta)$ and $t(\zeta)$ are rational functions and $\kappa = 0$, or 1. It also contains a formula for the solution of

$$a(\zeta)\int_L \frac{\phi(z)dz}{z-\zeta}+\int_L \frac{b(z)\phi(z)dz}{z-\zeta}=f(\zeta)$$

if $a(\zeta)b(\zeta) = -1$; $a(\zeta) \neq 1, -1$.

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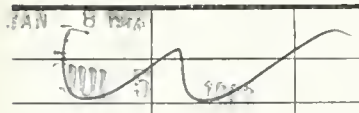

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